

Solution to the non-perturbative renormalization of gauge theory

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Abstract. The long standing problem of a non-perturbative renormalization of a gauge field theoretical Hamiltonian is addressed and explicitly carried out within an (effective) light-cone Hamiltonian approach to QCD. The procedure is in line with the conventional ideas: The Hamiltonian is first regulated by suitable cut-off functions, and subsequently renormalized by suitable counter terms to make it cut-off independent. Emphasized is the considerable freedom in the cut-off function which eventually can modify the Coulomb potential of two charges at sufficiently small distances. The approach provides new physical insight into the nature of gauge theory and the potential energy of QCD and QED at short distance.

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1 Introduction

One of the most interesting subjects of particle and nuclear physics is to understand the internal structure of hadrons. A powerful description of the hadronic structure is obtained through the light-cone wave functions. They encode all possible quark and gluon momenta, and helicity and flavor correlations in the hadron and are constructed from the QCD light-cone Hamiltonian [1]:

$$H_{LC} = P^+ P^- - P_\perp^2,$$

where $P^\pm = P^0 \pm P^z$. The wave function Ψ_b of a hadron b with mass M_b satisfies the relation [1]:

$$H_{LC}|\Psi_b\rangle = M_b^2|\Psi_b\rangle.$$

The light-cone wave functions are expanded in terms of a complete basis of Fock states having increasing complexity. In this way the hadron presents itself as an ensemble of coherent states containing various numbers of quark and gluon quanta. For example, the positive pion has the Fock expansion:

$$|\Psi_{\pi^+}\rangle = \sum_n \langle n|\pi^+\rangle|n\rangle = \Psi_{u\bar{d}/\pi}^{(A)}(x_i, \mathbf{k}_{\perp i})|u\bar{d}\rangle + \Psi_{u\bar{d}g/\pi}^{(A)}(x_i, \mathbf{k}_{\perp i})|u\bar{d}g\rangle + \dots,$$

representing an expansion of the exact QCD eigenstate at scale A in terms of non-interacting quarks and gluons. The i_n particles in a Fock state (n) have longitudinal light-cone momentum fractions x_i and relative transverse momenta $\mathbf{k}_{\perp i}$, with

$$x_i = \frac{k_i^+}{P^+} = \frac{k_i^0 + k_i^z}{P^0 + P^z}, \quad \sum_{i=1}^{i_n} x_i = 1, \quad \sum_{i=1}^{i_n} \mathbf{k}_{\perp i} = \mathbf{0}_\perp.$$

The form of $\Psi_{n/H}(x_i, \mathbf{k}_{\perp i})$ is invariant under longitudinal and transverse boosts; i.e., the light-cone wave functions expressed in the relative coordinates x_i and $k_{\perp i}$ are independent of the total momentum (P^+ , \mathbf{P}_\perp) of the hadron. The first term in the expansion is referred to as the valence Fock state, as it relates to the hadronic description in the constituent quark model. The higher terms are related to the sea components of the hadronic structure. It has been shown that the remaining terms of the light-cone wave function can be determined once the valence Fock state is known [2,3]. Explicit expressions are given in [3].

When starting in 1984 with Discretized Light-Cone Quantization (DLCQ) [4] and with a revival of Dirac's Hamiltonian front form dynamics [5], all challenges of a gauge field Hamiltonian theory were essentially open questions, particularly

- the non-perturbative bound state problem,
- the many-body aspects of gauge field theory,
- regularization,
- renormalization,
- confinement,
- chirality,
- vacuum structure and condensates,

just to name a few. The step from the gauge field QCD Lagrangian down to a non-relativistic Schrödinger equation was completely mysterious. Now we know better [1].

We have understood, for example, that the chiral phase transition, in which the quarks are supposed to get their mass, is not the major challenge. The challenge is to understand what happens *after* the phase transition, at zero temperature.

The challenge is to understand the spectra of physical hadrons and to get the corresponding eigenfunctions, the light-cone wave functions.

We have learned how to partition the problem and how to shape our thinking in four major steps:

$$\begin{aligned} \mathcal{L}_{\text{QCD}} &\Downarrow \\ H_{\text{LC}}|\Psi\rangle &= M^2|\Psi\rangle \end{aligned} \quad (1)$$

$$\begin{aligned} &\Downarrow \\ H_{\text{eLC}}|\Psi_{q\bar{q}}\rangle &= M^2|\Psi_{q\bar{q}}\rangle \end{aligned} \quad (2)$$

$$\begin{aligned} &\Downarrow \\ H_{\text{eff}}|\Psi_{q\bar{q}}\rangle &= E|\Psi_{q\bar{q}}\rangle \end{aligned} \quad (3)$$

$$\begin{aligned} &\Downarrow \\ \left(\frac{\mathbf{p}^2}{2m_r} + V(\mathbf{x})\right)\psi(\mathbf{x}) &= E\psi(\mathbf{x}). \end{aligned} \quad (4)$$

The step from the Lagrangean \mathcal{L}_{QCD} to the light-cone Hamiltonian H_{LC} had been given in [1]; all its matrix elements are known and tabulated. Its reduction to an effective light-cone Hamiltonian H_{LC} has been given in [3,6] and will be sketched once again in Sec. 4. Some time will be spent in Sec. 5 to discuss the single particle Hamiltonian H_{eff} in momentum and in configuration space.

But first we must deal with the key issue of any gauge theory. We must say something about how to treat the logarithmic divergencies of the theory. We shall do it by conventional regularization and renormalization in Secs. 6 and 7. In the perturbative context of scattering theory the problem of regularization and renormalization was solved. In the non-perturbative context of a Hamiltonian it has not been solved. It is addressed in the next two sections.

Preliminary and condensed versions of this were given in conference proceedings [7]. Several typos and misprints are removed, also including some mistakes.

2 Regularization of the total Hamiltonian

Canonical field theory with the conventional QCD Lagrangian allows to derive the components of the total canonical four-momentum P^μ . Its front form version [1] rests on two assumptions, the light cone gauge $A^+ = 0$ [8] and the suppression of all zero modes [1,9]. The front form vacuum is then trivial.

I find it helpful to discuss the problem in terms of DLCQ [1,4]. In the back of my mind I visualize an explicit finite dimensional matrix representation of the Light-Cone Hamiltonian as it occurs for finite harmonic resolution. Such one is schematically displayed in Fig. 2 of [1]. All of its matrix elements are finite for any finite x and k_\perp .

The problem arises for ever increasing harmonic resolution, on the way to the continuum limit: The numerical eigenvalues are numerically unstable and diverge logarithmically [10,11], in contrast to the calculations in 1+1 dimension [4]; see also actual DLCQ calculations in 3+1 by Hiller [12].

The reason is inherent to Dirac's relativistic vertex interaction $\langle k_1, h_1 | V | k_2, h_2; k_3, h_3 \rangle$, in which some particle '1' is scattered into two particles '2' and '3' with their respective four-momenta k and helicities h , see Fig. 1. The

matrix element for bremsstrahlung, for example, is proportional to k_\perp , $\langle k_1, \uparrow | V | k_2, \uparrow; k_3, \uparrow \rangle \propto |\mathbf{k}_\perp|$, see Table 9 in [1], when the quark maintains its helicity while irradiating a gluon with four-momentum $k_3^\mu = (xP^+, \mathbf{k}_\perp, k_3^-)$. Singularities typically arise when over the squares of such matrix elements is integrated with respect to \mathbf{k}_\perp as in the integrations of perturbation theory.

As usual, regularization is not unique, and can be done in many ways. But not all regularization techniques of the past are applicable in a Hamiltonian approach. Dimensional regularization, for example, is not applicable in a matrix approach which is stuck with the precisely 3+1 dimensions of the physical world. Perturbative regularization [13] is not applicable in the non-perturbative context. The Fock space regularization of Lepage and Brodsky [8], see also [1], has blocked renormalization for many years. It was impossible to find suitable counter terms. In recent work, the invariant mass squared regularization has been abandoned in favor of Pauli-Villars regularization [12,14]. But thus far it is unclear how the approach [14] is applicable to the spectra of physical mesons. After applauding the light-cone approach [15], Wilson and collaborators [16] have attempted to base their considerations almost entirely on a renormalization group analysis, but no specific technology has emerged thus far.

In recent years I have come to favor another regularization, *vertex regularization*, which allows for renormalization and which is sufficiently simple on the technical level to be carried out explicitly.

The singularities are avoided *a priori* by vertex regularization, by multiplying each (typically off-diagonal) matrix element with a regulating *form factor* F :

$$\begin{aligned} \langle k_1, h_1 | V | k_2, h_2; k_3, h_3 \rangle &\Longrightarrow \\ \langle k_1, h_1 | V | k_2, h_2; k_3, h_3 \rangle F(Q) &. \end{aligned} \quad (5)$$

It took me several years to realize that it is the Feynman four-momentum transfer, $Q^2 = -(k_1 - k_2)^2$, across a vertex, which governs an effective interaction. The minimal requirement for such a universal form factor is then

$$F(Q) = F(Q; \Lambda) = \begin{cases} 1, & \text{for } Q^2 \rightarrow 0, \\ 0, & \text{for } Q^2 \rightarrow \infty. \end{cases} \quad (6)$$

It is satisfied by the step function, $F(Q) = \Theta(Q^2 - \Lambda^2)$. The limit $\Lambda \rightarrow 0$ suppresses the interaction all together, the limit $\Lambda \rightarrow \infty$ restores the interaction and its problems. Any finite value of Λ^2 restricts Q^2 to be finite and eliminates the singularities. But the sharp cut-off generates problems in another corner of the theory and $F(Q)$ must be an analytic function of Q , as to be seen below.

Vertex regularization takes thus care of the ultraviolet divergences. The (light-cone) infrared singularities are taken care of as usual by a kinematical gluon mass.

Emphasized should be that *vertex regularization* does not chop off individual Fock space constituents, nor does it chop off Fock space components with large momenta, as in Fock-space regularization [1,8]. Possible problems with gauge invariance are thus less severe and unlikely. Spoken in terms of matrices, the dimension of the Hamiltonian

Fig. 1.

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matrix can be as large as it wants to be. The limit $\Lambda \rightarrow 0$ is represented by a diagonal matrix.

3 Renormalization of the total Hamiltonian

The non-perturbative renormalization of the Hamiltonian was stuck for many years by the fact that the coupling constant g multiplies the regulator function $F(Q)$ in Eq.(5). It has always been clear that one may add *non local counter terms* [16], but it was not clear how they could be constructed. An explicit construction is given below.

What follows is the general but abstract procedure of modern renormalization theory [17].

Suppose to have solved the problem for a fixed value of the 7 ‘bare’ parameters in the Lagrangian, the coupling constant $g = g_0$ and the 6 flavor quark masses $m_f = (m_f)_0$, and for a fixed value of the exterior cut-off scale $\Lambda = \Lambda_0$. Suppose further that these 7+1 parameters are determined such, that the calculated mass squared eigenvalues M_i^2 agree with the corresponding experimental numbers. Next, suppose to change the cut-off by a small amount $\delta\Lambda$. Every calculated eigenvalue will then change by δM_i^2 . Renormalization theory is then the attempt to reformulate the Hamiltonian, such, that all changes δM_i^2 vanish identically.

The fundamental renormalization group equation is therefore:

$$dM_i^2|_0 = dM_i^2|_{g=g_0, m_f=m_{f_0}, \Lambda=\Lambda_0} = 0, \quad (7)$$

for all eigenstates i . Equivalently one requires that the Hamiltonian is stationary with respect to small $\delta\Lambda$:

$$\delta H_{LC}|_0 = 0. \quad (8)$$

Hence forward reference to $(g_0, m_{f_0}, \Lambda_0)$, to the ‘renormalization point’, will be suppressed.

Next, I specify a non local counter term, as follows. To the original Hamiltonian one adds a counter term Hamiltonian, whose interaction has exactly the same structure except that the regulator function $F(Q)$ is replaced by an other function $\bar{F}(Q)$, that is by

$$F(Q, \Lambda) \implies \bar{F}(Q, \Lambda) = F(Q, \Lambda) + C(Q, \Lambda). \quad (9)$$

Here, the regulating function $F(Q) = F(Q, \Lambda)$ is supposed to be a given function. The counter term $C(Q) = C(Q, \Lambda)$ is unknown, subject to the constraint that it vanishes at the renormalization point,

$$C(Q, \Lambda) \Big|_{\Lambda=\Lambda_0} = 0. \quad (10)$$

The goal is then to find a $C(Q, \Lambda)$ such that Eq.(8) is satisfied identically.

According to Eq.(5) the light-cone Hamiltonian depends on Λ only through the function $\bar{F}(Q; \Lambda)$. Its variation with respect to Λ is therefore

$$\delta H_{LC} = \delta \bar{F} \frac{\delta H_{LC}}{\delta \bar{F}}.$$

Eq.(7) as the fundamental equation of renormalization theory can therefore be replaced by

$$\delta \bar{F} = \delta \Lambda \frac{\partial \bar{F}}{\partial \Lambda} = 0, \quad (11)$$

since the variational derivative of the Hamiltonian with respect to the regulator cannot vanish identically.

The fundamental equation (11) defines then a differential equation for $C(\Lambda) = C(Q; \Lambda)$:

$$\frac{dC(Q; \Lambda)}{d\Lambda} = -\frac{dF(Q; \Lambda)}{d\Lambda}, \quad (12)$$

which, in its integral form, includes the initial condition

$$C(Q, \Lambda) = -\int_{\Lambda_0}^{\Lambda} ds \frac{dF(Q, s)}{ds} = F(Q, \Lambda_0) - F(Q, \Lambda). \quad (13)$$

The renormalized regulator function is *manifestly independent of Λ* :

$$\begin{aligned} \bar{F}(Q, \Lambda) &= F(Q, \Lambda) + C(Q, \Lambda) \\ &= F(Q, \Lambda) + (F(Q, \Lambda_0) - F(Q, \Lambda)) \\ &= F(Q, \Lambda_0). \end{aligned} \quad (14)$$

The counter term improved Hamiltonian is renormalization group invariant. By construction, the value of Λ_0 must be determined by experiment.

One should emphasize an important point: In deriving Eq.(14), it was assumed implicitly that the regulator function has well defined derivatives with respect to Λ . The theta function of the sharp cut-off, however, is a distribution with only ill defined derivatives; therefore, it must be removed from the class of admitted regulator functions.

Another point is equally important: The step function has only a single parameter, Λ . Renormalization theory admits *any function* $F(Q, \Lambda) = f(Q/\Lambda)$ as long as $f(x)$ has well defined derivatives with respect to $x = Q/\Lambda$.

For a long time, the main result of this section looked to me like sheer witchcraft: One selects a particular value of $\Lambda = \Lambda_0$, and then it turns out that the Hamiltonian is stationary with respect to small changes in Λ , for any value of Λ_0 !

How can one believe this strange result? Is it only formal? — The abstract procedure remains somewhat academic since a reliable numerical solution of the bound state equation, Eq.(1), would exhaust a large fraction of the worlds computer power, not to speak about the many variations which should be performed. To carry out such a programme would probably be impossible. Progress has come only from recent work on a particular model which did allow to formulate a paradigmatic example [18] in modern renormalization theory, see below.

The result is in sharp contrast to perturbative renormalization, where the renormalized quantity is *independent of the regularization scale*. Here, in the non-perturbative renormalization of a Hamiltonian, the regularization scale Λ_0 is fixed from the outset and becomes a member of the canonical parameters, like mass and coupling constant. It may not be driven to the extreme, to infinity or to zero, it must be kept at the finite value fixed *ab initio*.

4 The effective (light-cone) Hamiltonian

In a field theory, one is confronted with a many-body problem of the worst kind: Not even the particle number is conserved. In order to formulate effective Hamiltonians more systematically, a novel many-body technique had to be developed, the *method of iterated resolvents* [3, 6], whose details are not important here.

Important is that the method of iterated resolvents does not chop off Fock states or Fock state classes, and that it *does not violate on purpose* symmetries like gauge and Lorentz invariance.

Important is as well that the *effective light-cone Hamiltonian* H_{eLC} has the same eigenvalue as the *full light-cone Hamiltonian* H_{LC} and that it generates the bound state wave function of valence quarks by an one-body integral equation in (x, \mathbf{k}_\perp) [6]:

$$\begin{aligned} M^2 \psi_{h_1 h_2}(x, \mathbf{k}_\perp) &= \left[\frac{\bar{m}_1^2 + \mathbf{k}_\perp^2}{x} + \frac{\bar{m}_2^2 + \mathbf{k}_\perp^2}{1-x} \right] \psi_{h_1 h_2}(x, \mathbf{k}_\perp) \\ &- \frac{1}{3\pi^2} \sum_{h'_q, h'_q} \int \frac{dx' d^2 \mathbf{k}'_\perp}{\sqrt{x(1-x)x'(1-x')}} \psi_{h'_1 h'_2}(x', \mathbf{k}'_\perp) \\ &\times F(Q_q) F(Q_{\bar{q}}) \left(\frac{\bar{\alpha}(Q_q)}{2Q_q^2} + \frac{\bar{\alpha}(Q_{\bar{q}})}{2Q_{\bar{q}}^2} \right) \\ &\times [\bar{u}(k_1, h_1) \gamma^\mu u(k'_1, h'_1)] [\bar{v}(k'_2, h'_2) \gamma_\mu v(k_2, h_2)] . \quad (15) \end{aligned}$$

One has arrived at Eq.(2): $M^2 |\Psi_{q\bar{q}}\rangle = H_{\text{eLC}} |\Psi_{q\bar{q}}\rangle$. Here, M^2 is the eigenvalue of the invariant-mass squared. The associated eigenfunction $\psi_{h_1 h_2}(x, \mathbf{k}_\perp)$ is the probability amplitude $\langle x, \mathbf{k}_\perp, h_1; 1-x, -\mathbf{k}_\perp, h_2 | \Psi_{q\bar{q}} \rangle$ for finding the quark with momentum fraction x , transversal momentum \mathbf{k}_\perp and helicity h_1 , and correspondingly the anti-quark. Expressions for the (effective) quark masses \bar{m}_1 and \bar{m}_2 and the (effective) coupling function $\bar{\alpha}(Q)$ are given in [6]. Q_q and $Q_{\bar{q}}$ are the Feynman momentum transfers of quark and anti-quark, respectively, and $u(k_1, h_1)$ and $v(k_2, h_2)$ are their Dirac spinors in the Lepage-Brodsky convention [8], given explicitly in [1]. They are arranged in the Lorenz scalar spinor matrix

$$\begin{aligned} \langle h_1, h_2 | S | h'_1, h'_2 \rangle &= [\bar{u}(k_1, h_1) \gamma^\mu u(k'_1, h'_1)] \\ &\times [\bar{v}(k'_2, h'_2) \gamma_\mu v(k_2, h_2)] \quad (16) \end{aligned}$$

which is a rather complicated (matrix) function of its six arguments $x, x', \mathbf{k}_\perp, \mathbf{k}'_\perp$, as tabulated in [19]. Finally, the form factors $F(Q)$ restrict the range of integration and regulate the interaction. Note that the equation is fully relativistic and covariant.

It should be emphasized that Eq.(15) is valid only for quark and anti-quark having different flavors [3, 6]. The additional annihilation term for identical flavors is omitted, but presently being investigated [20]. It should also be emphasized that the same structure was obtained by a completely different method, with Wegner's Hamiltonian flow equations [21]. In [21] it is also shown why the concept of a 'mean momentum transfer', $Q^2 = \frac{1}{2} (Q_q^2 + Q_{\bar{q}}^2)$ is a meaningful simplification. It allows to replace Eq.(15)

by

$$\begin{aligned} M^2 \psi_{h_1 h_2}(x, \mathbf{k}_\perp) &= \left[\frac{\bar{m}_1^2 + \mathbf{k}_\perp^2}{x} + \frac{\bar{m}_2^2 + \mathbf{k}_\perp^2}{1-x} \right] \psi_{h_1 h_2}(x, \mathbf{k}_\perp) \\ &- \frac{1}{3\pi^2} \sum_{h'_q, h'_q} \int \frac{dx' d^2 \mathbf{k}'_\perp}{\sqrt{x(1-x)x'(1-x')}} \frac{\bar{\alpha}(Q)}{Q^2} R(Q) \\ &\times [\bar{u}(k_1, h_1) \gamma^\mu u(k'_1, h'_1)] [\bar{v}(k'_2, h'_2) \gamma_\mu v(k_2, h_2)] . \quad (17) \end{aligned}$$

The form factors $F(Q)$ have made their way into the regulator function

$$R(Q) = F^2(Q) . \quad (18)$$

The same bilinear combination occurs also in the explicit expressions for \bar{m} and $\bar{\alpha}(Q)$ [6], such that one can replace the $F(Q)$'s by $R(Q)$ as an independent variable.

It is a light cone peculiarity that the (light-cone) Hamiltonian H_{LC} is additive in kinetic energy terms and the interaction [1]. The same holds for the effective Hamiltonian H_{eff} . Since it is allowed to subtract a c-number from H_{eLC} and to divide the result by another c-number, one can define an effective Hamiltonian H_{eff} , implicitly by

$$H_{\text{eLC}} \equiv (\bar{m}_1 + \bar{m}_2)^2 + 2(\bar{m}_1 + \bar{m}_2) H_{\text{eff}} , \quad (19)$$

and explicitly by

$$H_{\text{eff}} = \frac{H_{\text{eLC}} - (\bar{m}_1 + \bar{m}_2)^2}{2(\bar{m}_1 + \bar{m}_2)} . \quad (20)$$

Such a manipulation does not change the eigenfunctions.

The eigenvalues then have the dimension of an energy

$$H_{\text{eff}} |\Psi_{q\bar{q}}\rangle = E |\Psi_{q\bar{q}}\rangle ,$$

arriving this way at Eq.(3). As will be seen below, H_{eff} has much in common with a conventional non-relativistic Hamiltonian in momentum representation, but the approach remains fully relativistic and covariant.

Note that mass and energy in the front form, on the light cone, are related by

$$M^2 = (\bar{m}_1 + \bar{m}_2)^2 + 2(\bar{m}_1 + \bar{m}_2) E , \quad (21)$$

and **not** by

$$\begin{aligned} M^2 &= (\bar{m}_1 + \bar{m}_2 + E)^2 \\ &= (\bar{m}_1 + \bar{m}_2)^2 + 2(\bar{m}_1 + \bar{m}_2) E + E^2 \\ &\neq (\bar{m}_1 + \bar{m}_2)^2 + 2(\bar{m}_1 + \bar{m}_2) E , \quad (22) \end{aligned}$$

as usual in the instant form which quantizes the system at equal usual time. Only if the energy is negligible, *i.e.* only if $(E/(\bar{m}_1 + \bar{m}_2))^2 \ll 1$, the two relations coincide.

4.1 The effective Hamiltonian in different clothes

A rather drastic technical simplification is achieved by a transformation of the integration variable. One can substitute the integration variable x by the integration variable

k_z , which, for all practical purposes, can be interpreted [1] as the z -component of a 3-momentum vector $\mathbf{p} = (k_z, \mathbf{k}_\perp)$. For equal masses $\overline{m}_1 = \overline{m}_2 = m$, the transformation is, together with its inverse,

$$x(k_z) = \frac{1}{2} \left[1 + \frac{k_z}{\sqrt{m^2 + \mathbf{k}_\perp^2 + k_z^2}} \right]. \quad (23)$$

$$k_z^2(x) = (m^2 + \mathbf{k}_\perp^2) \frac{(x - \frac{1}{2})^2}{x(1-x)}. \quad (24)$$

Inserting these substitutions into Eq.(17) and defining the reduced wave function $\varphi_{h_1 h_2}$ by [19]

$$\psi_{h_1 h_2}(x, \mathbf{k}_\perp) = \frac{\sqrt{A(k_z, \mathbf{k}_\perp)}}{\sqrt{x(1-x)}} \varphi_{h_1 h_2}(k_z, \mathbf{k}_\perp), \quad (25)$$

$$A(\mathbf{p}) = \sqrt{1 + \frac{\mathbf{p}^2}{m^2}}, \quad (26)$$

leads to an integral equation in 3-momentum space, in which all reference to light-cone variables has disappeared:

$$\begin{aligned} E \varphi_{h_1 h_2}(\mathbf{p}) &= \frac{\mathbf{p}^2}{2m_r} \varphi_{h_1 h_2}(\mathbf{p}) \\ &- \frac{1}{2\pi^2} \sum_{h'_1, h'_2} \int d^3 p' \frac{\varphi_{h'_1 h'_2}(\mathbf{p}')}{\sqrt{A(p)A(p')}} \frac{\overline{\alpha}_c(Q)}{Q^2} \frac{R(Q)}{4m^2} \\ &\times [\overline{u}(k_1, h_1) \gamma^\mu u(k'_1, h'_1)] [\overline{v}(k'_2, h'_2) \gamma_\mu v(k_2, h_2)], \end{aligned} \quad (27)$$

with $\overline{\alpha}_c(Q) = \frac{4}{3} \overline{\alpha}(Q)$. The reduced mass for $\overline{m}_1 = \overline{m}_2 = m$ is $m_r = m/2$. Its Fourier transform gives Eq.(4).

5 Two over-simplified models

Krautgärtner *et al* [10] and Trittman *et al* [11] have shown how to solve an equation like Eq.(17) numerically with high precision. But since the numerical effort is considerable, it is reasonable to work first with an over simplified model, as specified next.

The Singlet-Triplet model. Quarks are at relative rest when $\mathbf{k}_\perp = 0$ and $x = \overline{x}$, with $\overline{x} \equiv \overline{m}_1/(\overline{m}_1 + \overline{m}_2)$. An inspection of Eq.(33) in [19] reveals that for very small deviations from the equilibrium values, the spinor matrix $\langle h_1, h_2 | S | h'_1, h'_2 \rangle$ is proportional to the unit matrix,

$$\langle h_1, h_2 | S | h'_1, h'_2 \rangle \simeq 4\overline{m}_1 \overline{m}_2 \delta_{h_1, h'_1} \delta_{h_2, h'_2}. \quad (28)$$

For very large deviations, particularly for $\mathbf{k}_\perp'^2 \gg \mathbf{k}_\perp^2$, holds

$$Q^2 \simeq \mathbf{k}_\perp'^2, \quad \text{and} \quad \langle \uparrow \downarrow | S | \uparrow \downarrow \rangle \simeq 2\mathbf{k}_\perp'^2. \quad (29)$$

The *Singlet-Triplet (ST) model* combines these aspects:

$$\langle h_1, h_2 | S | h'_1, h'_2 \rangle = \delta_{h_1, h'_1} \delta_{h_2, h'_2} \langle h_1, h_2 | S | h_1, h_2 \rangle, \quad (30)$$

$$\frac{\langle h_1, h_2 | S | h_1, h_2 \rangle}{Q^2} = \begin{cases} \frac{4\overline{m}_1 \overline{m}_2}{Q^2} + 2, & \text{for } h_1 = -h_2, \\ \frac{4\overline{m}_1 \overline{m}_2}{Q^2}, & \text{for } h_1 = h_2. \end{cases} \quad (31)$$

For anti parallel helicities $h_1 = -h_2$ (singlet) the model interpolates between two extremes: For small momentum transfer Q , the ‘2’ in Eq.(29) is unimportant and the Coulomb aspects of the first term prevail. For large Q , the Coulomb aspects are unimportant and the hyperfine interaction is dominant. The ‘2’ carries the singlet-triplet mass difference.

For parallel helicities $h_1 = h_2$ (triplets) the model reduces to the Coulomb kernel. The model over emphasizes many aspects but its simplicity has proven useful for quick and analytical calculations. Most importantly, the model allows to drop the helicity summations, simplifying the problem enormously on the technical level.

The model can not be justified in the sense of an approximation, but it emphasizes the point that the ‘2’, or any other constant in the kernel of an integral equation, leads to singularities and thus to numerically undefined equations. Replacing the function $\overline{\alpha}(Q)$ by the strong coupling constant $\alpha_s = g^2/4\pi$ completes the model assumptions.

For the singlet, the ST-model translates Eq.(27) identically into

$$\begin{aligned} E \varphi(\mathbf{p}) &= \frac{\mathbf{p}^2}{2m_r} \varphi(\mathbf{p}) \\ &- \frac{\alpha_c}{2\pi^2} \int \frac{d^3 p'}{\sqrt{A(p)A(p')}} \left(\frac{4m^2}{Q^2} + 2 \right) \frac{R(Q)}{4m^2} \varphi(\mathbf{p}'), \end{aligned} \quad (32)$$

with $\alpha_c = \frac{4}{3} \alpha_s$. The equation for the triplets is obtained by dropping the ‘2’. The first term in this equation, $\mathbf{p}^2/2m_r$, coincides with the kinetic energy in a conventional non-relativistic Hamiltonian. This is remarkable in view of the fact that no approximation to this extent has been made. The fully relativistic and covariant light-cone approach has no relativistic corrections in the kinetic energy!

Since the first term in Eq.(32) is a kinetic energy, the second must be a potential energy — in momentum representation. In principle, it could be Fourier transformed with $e^{-i\mathbf{p}\mathbf{r}}$ to configuration space with the variable \mathbf{r} . But due to the factor $A(p)A(p')$ in the kernel, the resulting potential energy would be non-local. A Fourier transformation with \mathbf{Q} , *i.e.* with $e^{-i\mathbf{Q}\mathbf{r}}$, would not make much sense to me.

To avoid this non-locality, I introduce the ‘Fourier simplification’

$$A(p) \equiv 1. \quad (33)$$

With $A(p) = 1$, the mean four momentum transfer Q^2 reduces to the three momentum transfer $q^2 = (\mathbf{p} - \mathbf{p}')^2$. In consequence, the kernel of Eq.(32),

$$U(\mathbf{q}) = -\frac{\alpha}{2\pi^2} \left(\frac{4m^2}{q^2} + 2 \right) \frac{R(q)}{4m^2}, \quad (34)$$

depends only on $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. Its Fourier transform is a local function,

$$V(\mathbf{r}) = \int d^3 q e^{-i\mathbf{q}\mathbf{r}} U(\mathbf{q}), \quad (35)$$

playing the role of a conventional *potential energy* in the Fourier transform of Eq.(32), *i.e.* in

$$E \psi(\mathbf{r}) = \left[\frac{\mathbf{p}^2}{2m_r} + V(\mathbf{r}) \right] \psi(\mathbf{r}) . \quad (36)$$

Here is the Schrödinger equation from Eq.(4) in full glory! Despite its conventional structure it is a front form equation, designed to calculate the light-cone wave function $\psi(\mathbf{r}) \rightarrow \varphi(\mathbf{p}) \rightarrow \psi_{q\bar{q}}(x, \mathbf{k}_\perp)$.

The $\uparrow\downarrow$ -model. In an early stage of the present work, I thought that the Coulomb term in Eq.(34) needs no regularization. In consequence I replaced the kernel by

$$U(\mathbf{q}) = -\frac{\alpha}{2\pi^2} \left(\frac{4m^2}{q^2} + 2R(q) \right) \frac{1}{4m^2} , \quad (37)$$

referred to as the $\uparrow\downarrow$ -model.

In view of the present insight, see [24], this misses the point completely and is completely false. Nevertheless, the model has played an important role in developing the present renormalization theory.

6 Paradigms for renormalization theory

In this section, I apply the $\uparrow\downarrow$ -model to the the general procedure as outlined in Sect. 3. It is simple enough to go to every single aspect *in concreto*. In particular, I can solve the eigenvalue problem, $H_{\text{eLC}}|\varphi_i\rangle = M_i^2|\varphi_i\rangle$.

The lowest eigenvalue M_0^2 agrees with the pion mass at the given values of

$$\begin{aligned} \alpha &= \alpha_0 = 0.69 , \\ m &= m_0 = 1.16\Delta , \\ \Lambda &= \Lambda_0 = 3.80\Delta , \end{aligned} \quad (38)$$

with the mass unit $\Delta = 350$ MeV. The triple will be called the renormalization point.

When changing the parameters α and Λ (at a fixed value of $m = m_0$), all eigenvalues M_i^2 will change and be functions of (α, Λ) , among them the mass-squared eigenvalue for the ground state, $M_0^2(\alpha, \Lambda)$. This function is displayed as a contour plot in Fig. 2(left). Note that this plot is not schematic but the result of an actual calculation. Calculated with 100 meshpoints in each direction, the figure represents some 10^4 solutions of the eigenvalue problem. It goes without saying that such can be done only for a sufficiently simple model, the $\uparrow\downarrow$ -model, with a sufficiently simple soft regulator

$$R(Q; \Lambda) = \bar{R}_0(Q; \Lambda) = \frac{\Lambda^2}{\Lambda^2 + Q^2} , \quad (39)$$

and with sufficiently fast numerical procedures.

According to the general outline in Sec. 3, one must make sure that the invariant mass squared *spectrum* stays invariant, $\delta M_i^2(\alpha, \Lambda) = 0$, for infinitesimal variations $\delta\Lambda$. I do that by construction, by introducing a *new regulator*

$$\bar{R}(Q, \Lambda) = R(Q, \Lambda) - (\Lambda^2 - \Lambda_0^2) \frac{\partial R(Q, \Lambda)}{\partial \Lambda^2} , \quad (40)$$

with the soft regulator $R(Q, \Lambda)$ given by Eq.(39).

Both \bar{R} and R have the same dimension. Both are in the class of admitted functions, see Eq.(6). Both are equal at the renormalization point $\Lambda = \Lambda_0$:

$$\bar{R}(Q, \Lambda_0) = R(Q, \Lambda_0) . \quad (41)$$

The derivative of \bar{R} vanishes at the renormalization point by construction, since

$$\frac{\partial \bar{R}(Q, \Lambda)}{\partial \Lambda^2} = 2Q^2 \frac{(\Lambda^2 - \Lambda_0^2)}{(\Lambda^2 + Q^2)^3} . \quad (42)$$

This derivative is certainly not the zero function for all Λ , and renormalization can be valid at most locally, in the vicinity of the renormalization point.

Based on the Hellmann-Feynman theorem, one expects a vanishing derivative of the eigenvalue with respect to Λ near the renormalization point. The actual numerical results in Fig. 2(right) illustrate this very convincingly: All contours have zero derivatives at $\Lambda = \Lambda_0$. The Hamiltonian is locally renormalized: $\delta M_i^2(\alpha, \Lambda) \sim 0$.

6.1 Renormalization by a sequence of regulators

With $\bar{R}_0(Q, \Lambda)$ given by Eq.(39), the above can be put into a systematic form. I can define a whole series of regulator functions $\bar{R}_n(Q, \Lambda)$ by

$$\begin{aligned} \bar{R}_1(Q, \Lambda) &= \bar{R}_0(Q, \Lambda) - Q^2 \frac{(\Lambda^2 - \Lambda_0^2)}{(\Lambda^2 + Q^2)^2} , \\ \bar{R}_2(Q, \Lambda) &= \bar{R}_1(Q, \Lambda) - Q^2 \frac{(\Lambda^2 - \Lambda_0^2)^2}{(\Lambda^2 + Q^2)^3} , \\ &\dots \\ \bar{R}_n(Q, \Lambda) &= \bar{R}_{n-1}(Q, \Lambda) - Q^2 \frac{(\Lambda^2 - \Lambda_0^2)^n}{(\Lambda^2 + Q^2)^{n+1}} . \end{aligned} \quad (43)$$

Each member is a genuine regulator satisfying the minimal requirements of Eq.(6). All are different from the original regulator in Eq.(39), and none can be preferred over the other. Their derivatives given by

$$\begin{aligned} \frac{d\bar{R}_1(Q, \Lambda)}{d\Lambda^2} &= 2Q^2 \frac{(\Lambda^2 - \Lambda_0^2)}{(\Lambda^2 + Q^2)^3} , \\ \frac{d\bar{R}_2(Q, \Lambda)}{d\Lambda^2} &= 3Q^2 \frac{(\Lambda^2 - \Lambda_0^2)^2}{(\Lambda^2 + Q^2)^4} , \\ &\dots \\ \frac{d\bar{R}_n(Q, \Lambda)}{d\Lambda^2} &= (n+1)Q^2 \frac{(\Lambda^2 - \Lambda_0^2)^n}{(\Lambda^2 + Q^2)^{n+2}} , \end{aligned}$$

they all are stationary at the renormalization point. Carrying on the procedure to higher and higher orders, the contours corresponding to Fig. 2(right) become broader and broader. In the limit of large order the contours become flat: One has realized the fundamental renormalization group equation: $dM_i^2(\alpha, \Lambda) = 0$.

One can show this even analytically by considering the limit $n \rightarrow \infty$. With

$$x = \frac{\Lambda^2 - \Lambda_0^2}{\Lambda^2 + Q^2} , \quad (44)$$

Fig. 2. Con-
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the regulators $\bar{R}_n(Q, \Lambda)$ behave in the limit $n \rightarrow \infty$ like

$$\begin{aligned}\bar{R}_n(Q, \Lambda) &= \bar{R}_0(Q, \Lambda) - Q^2 \frac{\Lambda^2 - \Lambda_0^2}{\Lambda^2 + Q^2} [1 + x + x^2 + \dots] \\ &= \frac{\Lambda^2}{\Lambda^2 + Q^2} - Q^2 \frac{\Lambda^2 - \Lambda_0^2}{\Lambda^2 + Q^2} \frac{1}{1 - x} \\ &= \frac{\Lambda^2}{\Lambda^2 + Q^2} + \frac{\Lambda_0^2}{\Lambda_0^2 + Q^2} - \frac{\Lambda^2}{\Lambda^2 + Q^2} \\ &= \frac{\Lambda_0^2}{\Lambda_0^2 + Q^2},\end{aligned}\quad (45)$$

which is *manifestly independent of Λ* . This result represents a beautiful and pedagogic example for how the renormalization group works.

6.2 Renormalization by counter terms

The regularization by counter terms has been discussed in Sec. 3. I repeat it here for clarity. Introduce a counter term Hamiltonian by

$$R(Q, \Lambda) \Rightarrow \bar{R}(Q, \Lambda) = R(Q, \Lambda) + C(Q, \Lambda). \quad (46)$$

The counter term $C(Q, \Lambda)$ is the unknown, subject to the constraint

$$C(Q, \Lambda) \Big|_{\Lambda=\Lambda_0} = 0. \quad (47)$$

The renormalization condition,

$$\frac{d\bar{R}(Q, \Lambda)}{d\Lambda} = 0, \quad (48)$$

defines the derivative of C :

$$\frac{dC(Q; \Lambda)}{d\Lambda} = -\frac{dR(Q; \Lambda)}{d\Lambda}. \quad (49)$$

Including the constraint by the integral form,

$$C(Q, \Lambda) = -\int_{\Lambda_0}^{\Lambda} ds \frac{dR(Q, s)}{ds} = R(Q, \Lambda_0) - R(Q, \Lambda), \quad (50)$$

determines the counter term. The renormalized regulator

$$\bar{R}(Q, \Lambda) = R(Q, \Lambda_0) \quad (51)$$

is then *manifestly independent of Λ* , in analogy to Eq.(14).

6.3 Renormalization by the subtraction method

After my talk at one of the Trento meetings, Tobias Frederico approached me with the proposal to put $R(q) = 1$ in Eq.(34), *i.e.*

$$U(\mathbf{q}) = -\frac{\alpha}{2\pi^2} \left(\frac{4m^2}{q^2} + 2 \right) \frac{1}{4m^2}, \quad (52)$$

take the Fourier transform according to Eq.(35), and define this way precisely the ill-defined Hamiltonian of the ‘Coulomb plus delta’ interaction according to Eq.(36), *i.e.*

$$H = \frac{\mathbf{p}^2}{2m_r} - \frac{\alpha}{r} + \mu\delta^{(3)}(\mathbf{r}). \quad (53)$$

Such a problem could then be solved by his ‘subtraction method,’ see *f.e.* Frederico *et al* [22], a method based on a T-matrix approach like in scattering theory.

Frederico’s initiative has put us into “the unique position to compare two drastically different schemes, both conceptually and numerically, and show that they agree” (citation from [22]).

This is of course no proof, but: ‘A reasonable man is satisfied with evidence, only a fool needs proof,’ as the famous mathematician Marc Kac has said in the context of trying to prove Boltzman’s H-theorem.

The step to the ‘oscillator plus delta’ interaction [23],

$$H = \frac{\mathbf{p}^2}{2m_r} - a + \frac{f}{2} r^2 + \mu\delta^{(3)}(\mathbf{r}), \quad (54)$$

was then straightforward and gave beautiful agreement with all of the the experimental *meson spectra*.

7 Perturbative versus non-perturbative renormalization

Perturbative renormalization addresses to render quantities of physical interest *independent of the regularization scale*. Conceptually, the scale can then be driven to infinity in the remainder of the theory, $\Lambda \Rightarrow \infty$, such that it drops out from the theory. Perturbative renormalization is adequate for scatterings problems.

Non-perturbative renormalization says: ‘No, don’t do that. Do not drive Λ to infinity.’ Set the universal renormalization scale Λ_0 in accord with experiment and give it the same rank as the other physical parameters of the theory, the coupling constant α and the mass m . Non-perturbative renormalization is adequate for bound state problems.

8 Summary and Conclusions

This work is an important mile stone on the long way from the canonical Lagrangian for quantum chromo dynamics down to the composition of physical hadrons in terms of their constituting quarks and gluons, by the eigenfunctions of a Hamiltonian.

As part of a on-going effort, a denumerable number of simplifying assumptions had to be phrased for getting a manageable formalism [3]. Among them is the formulation of an effective interaction by the method of iterated resolvents [6]. The strongest assumption in the present work, the simplifying Singlet-Triplet model in Sec. 4, is also the easiest to relax, as shown in the next paper.

The biggest progress of the present work can be found in Sects. 2 and 3. It is related to a consistent regularization and renormalization of a gauge theory. The ultraviolet divergences in gauge theory are caused less by the possibly large momenta of the constituent particles, but by the large momentum *transfers* in the interaction. In a Hamiltonian approach, such as the present, one has not much choice other than to chop them off by a regulating form factor in the elementary vertex interaction.

The form factor makes its way into a regulator function which suppresses the large momentum transfers in the Fourier transform of the Coulomb interaction. The arbitrariness in chopping off the *large momentum transfers* is reflected in the arbitrariness of the potential at *small short distances*. The arbitrariness allows for a pocket in the potential which binds the quarks in a hadron [24].

The problem is then how to fix this function with its many parameters, by experiment. In practice this is less difficult than anticipated. Actually, it has been done already [24]. It suffices to determine only three parameters, two continuous ones and one counting index.

The present work opens a broad avenue of applications, among them also the baryons and physical nuclei. But much work must be done in the future before such a simple approach be taken serious.

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